INTERNAL STRESSES IN MEDIA WITH MICROSTRUCTURE

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A large number of works (see [1 and 2], for example, where there are also references to other works) has been devoted to the description of a medium with microstructure within the framework of couple-stress elasticity theory. In [3 to 6] a theory has been developed, on the basis of crystal lattice theory, for a macroscopically homogeneous elastic medium with spatial dispersion which contains couple-stress elasticity theory as a particular case of weak dispersion. The general case of an inhomogeneous elastic medium with nonlocal interaction has been considered in [7].

A theory of internal stresses in a medium with microstructure, which is an extension of the continuum theory of dislocations, is considered herein. As is known, this latter has been constructed on the basis of a model of the elastic continuum and does not take account of the discrete structure of the medium, and in this sense, is an asymptotic theory of the interaction between defects at great distances. Taking account of the microstructure allows an essential broadening of the range of applicability of the theory and brings it closer to the theory of defects in a crystal lattice. The importance of such a generalization was mentioned in [6], in particular. The connection between dislocation theory and couple-stress elasticity theory of a Cosserat continuum has been considered in [11 to 16], and the motion of dislocations in a medium with spatial dispersion, in [17 and 18].

Section 1 herein is devoted to a general analysis of the statics of internal stresses in a medium with microstructure. The state of the medium is described by internal and external distortions, where the former is governed by the condition of orthogonality, with respect to the energy, of the external distortion tensor. The structure of the operator of the elastic moduli of the internal stresses is analyzed. It is shown that it generally differs from the operator of the elastic moduli of the external stresses by some operator which is localized in a domain where internal stress sources are concentrated. In this, and only in this domain may the internal stresses, in contrast to the external stresses, be nonsymmetric. The difference between the elastic moduli operators vanishes in the continuum approximation, and the stress tensor is symmetric everywhere. The Green's tensor for the internal stresses is determined, and its connection with the Green's tensor for the external stresses is established. Expressions are written down for the elastic energy and the stress fields of dislocations and of point defects in terms of the Green's tensor.

An explicit expression for the Green's tensor of internal stresses in an isotropic medium with spatial dispersion is constructed in Section 2. The asymptotic behavior of the Green's tensor is analyzed, and in particular, the contribution to the asymptotic from the nonlinear dependence of the dispersion curve on the wave vector is determined.

1. In constructing a theory of internal stresses in a medium with microstructure it is desirable, insofar as possible, to retain the general scheme and terminology of the continuum theory of dislocations. The basic difference between the model of an elastic medium with microstructure and the customary elastic continuum is the presence of a unit length element and of long-range forces, which results in the nonlocal character of the theory. This is reflected twofold in an appropriate formalism. Firstly, the field variables belong to a special class of entire analytic functions with a truncated Fourier spectrum [3]. Secondly, the elastic moduli are replaced by integral operators, where the kernels are not difference operators for a macroscopically inhomogeneous medium.

We shall assume that the state of stress of the medium is caused in the general case by both the effect of external forces, and the presence of internal stress sources. These sources may be, for instance, foreign atoms and vacancies, dislocations, grain boundaries, etc. In this case the variable field uniquely defining the state of the medium may naturally be considered an elastic distortion $\chi_{\alpha\beta}$, just as this is done in the continuum theory of dislocations [8]. The internal degrees of freedom (microrotations and microstrains) hence do not enter into the considerations explicitly. This is based on the fact that, as shown in [4 and 5], in the acoustic frequency domain, and therefore in the statics case also, the complex structure equations can be transformed into an equivalent simple structure by eliminating the internal degrees of freedom. Limiting ourselves to the harmonic approximation case, we write the most general expression for the elastic energy Φ of an infinite medium with microstructure as

$$2\Phi = \int\int \chi_{\alpha\beta}(x)\, S^{\alpha\beta\lambda\mu}(x, x')\, \chi_{\lambda\mu}(x')\, dx dx' = \frac{1}{(2\pi)^3}\int\int \overline{\chi_{\alpha\beta}(k)}\, S^{\alpha\beta\lambda\mu}(k, k')\, \chi_{\lambda\mu}(k')\, dk dk' \qquad (1.1)$$

Here x is a point of the medium, $\chi_{\alpha\beta}(k)$ is the Fourier transform of $\chi_{\alpha\beta}(x)$. From the assumption on the existence of a unit length element a in x-space, there follows that the base $\chi_{\alpha\beta}(k)$ is contained in some bounded domain of k-space with a characteristic dimension on the order of a^{-1}. The tensor $S^{\alpha\beta\lambda\mu}(x, x')$ is the kernel of the energy operator, and $S^{\alpha\beta\lambda\mu}(k,k')$ is its Fourier transform with respect to x and its Fourier original with respect to x'. Finally, the bar denotes the complex conjugate.

We shall henceforth write expressions of type (1.1) in the compact form

$$2\Phi = \langle \chi_{\alpha\beta} \,|\, S^{\alpha\beta\lambda\mu} \,|\, \chi_{\lambda\mu} \rangle \qquad (1.2)$$

The operator S is obviously Hermitian, i.e.,

$$S^{\alpha\beta\lambda\mu}(x, x') = S^{\lambda\mu\alpha\beta}(x', x), \qquad S^{\alpha\beta\lambda\mu}(k, k') = \overline{S^{\lambda\mu\alpha\beta}(k', k)} \qquad (1.3)$$

and it is also natural to consider it positive-definite.

From the requirement of invariance of the energy relative to rigid rotation of the medium we find just as in [3 and 7]

$$\langle a_{\alpha\beta} \,|\, S^{\alpha\beta\lambda\mu} \,|\, \chi_{\lambda\mu} \rangle = 0 \qquad (1.4)$$

for an arbitrary constant (in x-space) antisymmetric tensor $\chi_{\lambda\mu}$. Hence, the condition on $S(k, k')$

$$S^{[\alpha\beta]\lambda\mu}(0, k') = 0 \qquad (1.5)$$

follows from the arbitrariness of $\chi_{\lambda\mu}$.

The energy operator S defines the scalar product in the function space by transforming it into a Hilbert space. This latter may be decomposed into an orthogonal sum of external and internal distortion spaces, and the total energy is then represented as the sum of external and internal energies.

The external distortion $\zeta_{\alpha\beta}'$ is defined as the gradient of a displacement u_β caused by the external forces

$$\zeta_{\alpha\beta}'(x) = \partial_\alpha u_\beta(x), \qquad \mathrm{rot}^{\lambda\alpha}\zeta_{\alpha\beta}'(x) = 0 \tag{1.6}$$

We shall henceforth assume that the medium is macroscopically homogeneous at infinity. In this case, as has been shown in [7], the most general expression for the external energy may be written as follows:

$$2\Phi = \langle \zeta_{\alpha\beta}' \mid S^{\alpha\beta\lambda\mu} \mid \zeta_{\lambda\mu}' \rangle = \langle \varepsilon_{\alpha\beta}' \mid C^{\alpha\beta\lambda\mu} \mid \varepsilon_{\lambda\mu}' \rangle \tag{1.7}$$

Here $\varepsilon_{\alpha\beta}' = \zeta_{(\alpha\beta)}'$ is the strain tensor, and the Hermitian operator $C^{\alpha\beta\lambda\mu}$ is symmetric in the superscripts $\alpha\beta$ and $\lambda\mu$, and may be represented by virtue of the assumption made as

$$C^{\alpha\beta\lambda\mu}(x, x') = C_0^{\alpha\beta\lambda\mu}(x - x') + C_1^{\alpha\beta\lambda\mu}(x, x') \tag{1.8}$$

where $C_1(x, x') = 0$ for both $x \to \infty$ and $x' \to \infty$. For the external stresses $\sigma'^{\alpha\beta}$ we have

$$\sigma'^{\alpha\beta}(x) = \int C^{\alpha\beta\lambda\mu}(x, x')\,\varepsilon_{\lambda\mu}'(x')\,dx', \quad \partial_\alpha \sigma'^{\alpha\beta}(x) = -q^\beta(x) \tag{1.9}$$

Here the first relationship is the operator Hooke's law, and the second is the equilibrium equation whose right side contains the density of the volume forces $q^\beta(x)$.

The system of Eqs. (1.6), (1.9) and the conditions for the positive operator C at infinity define u, ζ' and σ' uniquely. In other words, there exists a Green's tensor G in terms of which the displacements are written thus

$$u_\alpha(x) = \int G_{\alpha\beta}(x, x')\, q^\beta(x')\,dx' \tag{1.10}$$

Representing the total distortion χ in the form $\chi = \zeta' + \zeta$, where ζ is the internal distortion, we require, by assumption, compliance with the orthogonality condition

$$\langle \zeta_{\alpha\beta}' \mid S^{\alpha\beta\lambda\mu} \mid \zeta_{\lambda\mu} \rangle = 0 \tag{1.11}$$

We then have for the internal energy

$$2\Phi = \langle \zeta_{\alpha\beta} \mid S^{\alpha\beta\lambda\mu} \mid \zeta_{\lambda\mu} \rangle \tag{1.12}$$

The quantity

$$\sigma^{\alpha\beta}(x) = \int S^{\alpha\beta\lambda\mu}(x, x')\,\zeta_{\lambda\mu}(x')\,dx' \tag{1.13}$$

evidently has the sense of an internal stress tensor. It follows from (1.11) that σ satisfies the Eq.

$$\partial_\alpha \sigma^{\alpha\beta}(x) = 0 \tag{1.14}$$

The right-hand side of Eq.

$$\mathrm{rot}^{\nu\lambda}\zeta_{\lambda\mu}(x) = \alpha^{\nu}_{.\mu}(x) \tag{1.15}$$

characterizes the density of internal stress sources. If dislocations are a physical internal stress source, then by definition α may be identified with the dislocation density, and in the general case, with the quasi-dislocation density in the Kröner terminology [8]. It is henceforth assumed that the internal stress sources are concentrated in a bounded domain of the space.

Eqs. (1.13) to (1.15) form a complete system of equations defining ζ and σ for a given dislocation density α.

Let us examine the structure of the operator S by assuming that the long-range is of the order of the unit length element a. Let us represent S as

$$S^{\alpha\beta\lambda\mu} = C^{\alpha\beta\lambda\mu} + T^{\alpha\beta\lambda\mu} \tag{1.16}$$

From physical considerations it follows that the Hermitian operator T should be localized, for a bounded long-range effect, in a domain where there are internal stress sources.

Hence, T should satisfy conditions resulting from (1.7) and (1.5)

$$\partial_\alpha \partial_\lambda' T^{\alpha\beta\lambda\mu}(x, x') = 0, \qquad T^{[\alpha\beta]\lambda\mu}(0, k') = 0 \tag{1.17}$$

It hence follows that the decomposition of the entire analytic function $T(k, k')$ in a series in k, k' starts with such a term

$$T^{\alpha\beta\lambda\mu}(k, k') = t^{\alpha\beta\lambda\mu\nu}k_\nu + t^{\lambda\mu\alpha\beta\nu}k_\nu' + \ldots \tag{1.18}$$

where $t^{\alpha\beta\lambda\mu\nu}$ is a constant tensor antisymmetric in the superscripts $\alpha\nu$ and symmetric in $\lambda\mu$. In an isotropic medium and in the presence of central symmetry the decomposition of $T(k, k')$ starts with a second order term in k, k'.

If $a \to 0$, i.e. the passage of the model of the customary elastic continuum is accomplished, then $T \to 0$ and S coincides with C. In this case the internal stress tensor is symmetric. In the general case $T \neq 0$, and the stress tensor is generally nonsymmetric but only in the domain where the internal stress sources are concentrated. For simplicity it is henceforth assumed that $S = C$. The $T \neq 0$ case may be considered analogously, but is associated with additional complications.

As is customary in continuum dislocation theory [8 and 10], it is convenient to introduce two other characteristics of the internal stress sources besides the dislocation density $\alpha(x)$: the incompatibility $\eta(x)$ and the density of the dislocation moments $m(x)$ by defining them thus

$$\eta^{\nu\rho}(x) = \mathrm{rot}^{(\rho|\mu|}\alpha^{\nu)}_{\cdot\mu}(x), \qquad \alpha^{\nu}_{\cdot\mu}(x) = \mathrm{rot}^{\nu\lambda} m_{\lambda\mu}(x) \tag{1.19}$$

Let us note that the density of the dislocation moments $m(x)$ is not defined uniquely [10]. Its base is generally broader than the base $\alpha(x)$ and contains the latter as a subset. Thus the base $\alpha(x)$ is a line of dislocations for a closed dislocation loop, and the base $m(x)$ is an arbitrary surface resting on the line of dislocations (*). It is evident from physical considerations that the base $\alpha(x)$ or $\eta(x)$ should be considered the true domain of concentration of the internal stress sources. The base $\alpha(x)$ and $m(x)$ coincide in the case of point effects. But in all cases the base, being given in an appropriate class of functions, turn out to be "smoothed out" by a quantity on the order of the unit length element a.

According to (1.19) the connection between $\eta(x)$ and $m(x)$ is given by

$$\eta^{\nu\rho}(x) = \mathrm{Rot}^{\nu\rho\lambda\mu} m_{\lambda\mu}(x) \tag{1.20}$$

where the operator Rot is defined by the expression (**)

$$\mathrm{Rot}^{\nu\rho\lambda\mu} = (\varepsilon^{\nu\varkappa\lambda}\varepsilon^{\rho\tau\mu}\partial_\varkappa\partial_\tau)_{(\lambda\mu)} \tag{1.21}$$

Let us write the solution of the system (1.13) to (1.15) as

$$\sigma^{\alpha\beta}(x) = \int F^{\alpha\beta\lambda\mu}(x, x') m_{\lambda\mu}(x')\, dx' \tag{1.22}$$

where $F^{\alpha\beta\lambda\mu}(x, x')$ is the Green's tensor for the internal stresses having the symmetry $C^{\alpha\beta\lambda\mu}(x, x')$.

Substitution of (1.22) into (1.13) to (1.15) yields an equation for F which in direct operator notation is

$$\mathrm{Rot}\, C^{-1}F = \mathrm{Rot} \tag{1.23}$$

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*) If a line current is compared to the dislocations in a magnetostatic analogy, then the density of the magnetic moments correspond to the density of the dislocation moments.
**) For convenience in the writing, the symmetrization operation is here included in the definition of Rot (see [8 and 10] also).

Applying the operator div to the right and taking into account that F is Hermitian, we find that $\mathrm{div} F = 0$. Hence, there follows representability of F as

$$F = \mathrm{Rot}\, H\, \mathrm{Rot} \tag{1.24}$$

where the operator H has the symmetry F, and is evidently not defined uniquely. It is expedient to utilize this nonuniqueness to obtain the simplest expression for H.

The internal elastic energy

$$2\Phi = \int \sigma^{\alpha\beta}(x)\, \varepsilon_{\alpha\beta}(x)\, dx \tag{1.25}$$

may now be written in the following three forms:

$$2\Phi = \langle m | F | m \rangle = \langle \alpha \mid \mathrm{rot}\, H\, \mathrm{rot} | \alpha \rangle = \langle \eta \mid H \mid \eta \rangle \tag{1.26}$$

The convenience of selecting any of the forms is hence governed by the specific form of the internal stress sources.

For example, let the point defect distribution

$$m_{\lambda\mu}(x) = v \sum_i M_{\lambda\mu}^{\ i}\, \delta(x - x_i) \tag{1.27}$$

be given, where $v = a^3$ and M^i is the nondimensional defect moment. In this case it is convenient to use F as the Green's tensor. We find directly

$$\sigma^{\alpha\beta}(x) = v \sum_i M_{\lambda\mu}^{\ i} F^{\alpha\beta\lambda\mu}(x, x_i), \quad 2\Phi = v^2 \sum_{ij} M_{\alpha\beta}^{\ i} M_{\lambda\mu}^{\ j} F^{\alpha\beta\lambda\mu}(x_i, x_j) \tag{1.28}$$

The density [10]

$$\alpha_{\cdot\mu}^{\nu}(x) = b_\mu \delta(L^\nu) \tag{1.29}$$

corresponds to a dislocation with contour L and Burger's vector b_μ.

We have for the stresses and energy (rot' acts on the argument x')

$$\begin{aligned} \sigma^{\alpha\beta}(x) &= b_\mu\, \mathrm{Rot}^{\alpha\beta\tau\rho} \int_L \mathrm{rot}'^{\mu\lambda} H_{\tau\rho\nu\lambda}(x, x_L')\, dL^\nu \\ 2\Phi &= b_\beta b_\mu \int_L \int_{L'} \mathrm{rot}^{\beta\rho}\, \mathrm{rot}'^{\mu\lambda} H_{\tau\rho\nu\lambda}(x_L, x_L')\, dL^\tau dL'^\nu \end{aligned} \tag{1.30}$$

It is interesting to establish a connection between the Green's tensors for the internal and external stresses. The method of equivalent force dipoles [10] may be used for this by comparing the density of the force dipole moments $m_{\lambda\mu}$ to the density of the dislocation moments $q^{\alpha\beta}$

$$q^{\alpha\beta}(x) = -\int C^{\alpha\beta\lambda\mu}(x, x')\, m_{\lambda\mu}(x')\, dx' \tag{1.31}$$

Omitting calculations analogous to those presented in [10], let us write the final result in operator form

$$F^{\alpha\beta\lambda\mu} = C^{\alpha\beta\lambda\mu} + C^{\alpha\beta\nu\rho} \nabla_\nu G_{\rho\tau} \nabla_\varkappa C^{\tau\varkappa\lambda\mu} \tag{1.32}$$

Therefore, F may be constructed if G is known. However, it will be shown below that in a number of cases it is more convenient to obtain explicit expressions for F and H by a direct method.

2. Let us consider a macroscopically homogeneous isotropic medium with spatial dispersion. In this case $C(x, x') = C(x - x')$ and in the k-representation

$$C^{\alpha\beta\lambda\mu}(k, k') = C^{\alpha\beta\lambda\mu}(k)\, \delta(k - k') \tag{2.1}$$

Here $C(k)$ is the Fourier transform of $C(x)$. For any isotropic medium

$$C(k) = 2\mu(k) E + \lambda(k) I \tag{2.2}$$

$$E^{\alpha\beta\lambda\mu} = {}^1/_2 (\delta^{\alpha\lambda}\delta^{\beta\mu} + \delta^{\alpha\mu}\delta^{\beta\lambda}), \quad I^{\alpha\beta\lambda\mu} = \delta^{\alpha\beta}\delta^{\lambda\mu} \tag{2.3}$$

where $\lambda(k)$, $\mu(k)$ are given functions of $k = |k_\alpha|$, in which k may be considered a bounded sphere with Debye radius $\varkappa = \pi/a$.

The tensor $B(k)$ inverse to $C(k)$ is

$$B(k) = \frac{1}{2\mu(k)} E - \frac{\lambda(k)}{2\mu(k)[3\lambda(k) + 2\mu(k)]} I \tag{2.4}$$

In this case the difference kernel $F(x - x')$, or in the k-representation, the kernel $F(k, k') = F(k)\,\delta(k - k')$ corresponds to the Green's operator F. Writing (1.24) in the k-representation, and replacing C^{-1} by $B(k)$ we obtain Eq. for $F(k)$

$$R^{\alpha\beta\nu\rho}(k)\, B_{\nu\rho\varkappa\tau}(k)\, F^{\varkappa\tau\lambda\mu}(k) = R^{\alpha\beta\lambda\mu}(k) \tag{2.5}$$

where $R(k)$ is the Fourier transform of the operator Rot

$$R^{\alpha\beta\lambda\mu}(k) = -(\varepsilon^{\alpha\rho\lambda}\varepsilon^{\beta\tau\mu}k_\rho k_\tau)_{(\lambda\mu)} \tag{2.6}$$

Rather cumbersome computations allow a representation of the solution of (2.5) as the following:

$$F^{\alpha\beta\lambda\mu}(k) = R^{\alpha\beta\nu\rho}(k)\, H_{\nu\rho\varkappa\tau}(k)\, R^{\varkappa\tau\lambda\mu}(k) \tag{2.7}$$

$$H(k) = 2\,\frac{\mu(k)}{k^4}\left[E + \frac{\lambda(k)}{\lambda(k) + 2\mu(k)} I\right] \tag{2.8}$$

To pass to the x-representation in (2.7) it is necessary to give explicit expressions for $\mu(k)$ and $\lambda(k)$. The functional dependence on k of the longitudinal ω_l and transverse ω_t free vibrations frequencies of the medium connected with λ , μ and the density ρ by means of the known relationships

$$\rho\omega_l^2(k) = k^2[\lambda(k) + 2\mu(k)], \qquad \rho\omega_t^2(k) = k^2\mu(k) \tag{2.9}$$

has a more graphic physical meaning.

From general considerations it follows that

$$\frac{d\omega_i(0)}{dk} = s_i, \qquad \frac{d\omega_i(\varkappa)}{dk} = 0 \tag{2.10}$$

where s_i is the speed of sound for $k = 0$, $i = l$, t . Different model approximations are know for the functions $\omega_i(k)$. Thus, it is assumed in the Born-Karman model that the $\omega_i(k)$ have the same dependence on k as in linear circuit theory, and in the Debye model the ω_i are considered linear functions of k, which is equivalent to the assumption $\lambda(k) = \lambda_0$, $\mu(k) = \mu_0$, where λ_0 , μ_0 are the usual Lamé constants. In the latter case the second of conditions (2.10) is certainly not satisfied. The simplest polynomial approximation with one orbitrary parameter γ_i is henceforth taken for $\omega_i^2(k)$:

$$\omega_i^2(k) = s_i^2k^2[1 - \gamma_i\varkappa^{-2}k^2 + {}^1/_3(2\gamma_i - 1)\varkappa^{-4}k^4] \tag{2.11}$$

which satisfies conditions (2.10). The parameter γ_i is connected uniquely with the boundary value of the frequency $\omega_i(\varkappa)$

$$\frac{\omega_i^2(\varkappa)}{s_i^2\varkappa^2} = \frac{2 - \gamma_i}{3} \tag{2.12}$$

Let us note that (2.11) practically corresponds to the Born-Karman model for $\gamma_i = 0.8$.

If the natural assumption is made that the boundary frequency is less than the Debye frequency, then the condition $-1 < \gamma_i < 2$ is imposed on γ_i .

Finally, let us assume that $\gamma_l = \gamma_t = \gamma$. This is equivalent to the assumption of independence of the Poisson coefficient ν from k. Then comparison of (2.11) with (2.9) yields

$$\mu(k) = \mu_0[1 - \gamma\varkappa^{-2}k^2 + {}^1/_3(2\gamma - 1)\varkappa^{-4}k^4] \tag{2.13}$$

and (2.8) becomes

$$H(k) = 2k^{-4}\mu(k)\left(E + \frac{\nu}{1-\nu} I\right) \tag{2.14}$$

To obtain $H(x)$ it is required to find the inverse Fourier transform for the function $h(k) = k^{-4}\mu(k)$ under the condition $k \leq \varkappa$. Calculations yield

$$h(r) = \mu(\Delta) f(r), \qquad r = |x| \tag{2.15}$$

$$f(r) = -\frac{1}{4\pi^2\varkappa}\left(\varkappa r \operatorname{Si} \varkappa r + \frac{\sin \varkappa r}{\varkappa r} + \cos \varkappa r\right) \tag{2.16}$$

$$\mu(\Delta) = \mu_0 [1 + \gamma\varkappa^{-2}\Delta + {}^1/_3 (2\gamma - 1)\varkappa^{-4}\Delta^2] \tag{2.17}$$

For the Debye model $\mu(\Delta) = \mu_0$. Let us also note Formulas

$$g(r) = \Delta f(r) = -\frac{1}{2\pi^2 r} \operatorname{Si} \varkappa r \tag{2.18}$$

$$\delta(x) = \Delta^2 f(r) = \frac{\varkappa}{2\pi^2 r^2}\left(\frac{\sin \varkappa r}{\varkappa r} - \cos \varkappa r\right) \tag{2.19}$$

where $\delta(x)$ plays the part of a three-dimensional δ-function, and $g(r)$ is the Green's function of the Laplace operator in the space of functions with a Fourier spectrum truncated by the Debye radius $\varkappa$

We now have for $H(x)$

$$H(x) = 2\left(E + \frac{\nu}{1-\nu} I\right) h(r) \tag{2.20}$$

Substitution into (1.24) yields an explicite expression for $F(x)$. This latter may also be transformed into

$$F(x) = \frac{2}{1-\nu} \operatorname{Rot} \operatorname{Rot} \mu(\Delta) f(r) + \frac{2\nu}{1-\nu} \operatorname{Rot} \mu(\Delta) g(r) \tag{2.21}$$

It is important to emphasize that the Green's tensors F and H are entire analytic functions, and therefore, have no singularities at $x = 0$. This permits writing expressions for F and H as well convergent series. From (2.15)to (2.17) we have for $h(r)$

$$h(r) = \frac{\mu_0}{6\pi^2\varkappa} \sum_{n=0}^{\infty} \frac{(-1)^n [2(2n+1)(2n+5) - \gamma(2n-1)(2n+7)]}{(4n^2-1)(2n+3)(2n+1)!} (\varkappa r)^{2n} \tag{2.22}$$

Using the expansion for $h(r)$, it is easy to find the value of $F(x)$ at $x = 0$, in particular, in terms of which the intrinsic energy of a point defect

$$F(0) = \frac{\beta\mu_0\varkappa^3}{45\pi^2(1-\nu)} [(7-5\nu)E + (1+5\nu)I] \tag{2.23}$$

is expressed according to (1.28), where $\beta \sim 1$ depends on the form of the function $\omega(k)$ and is determined, ultimately, by the coefficient of r^4 in the expansion of $h(r)$. We have for the Debye model and the approximation (2.11), respectively,

$$\beta = 1, \qquad \beta = \frac{30 - 11\gamma}{35} \tag{2.24}$$

Of special interest is the asymptotic behavior of the Green's tensor for large r, or equivalently, for $a \to 0$. It is easy to show that only the first two members in the expansions in (2.11) or (2.13) yield any contribution to the asymptotic values. Higher order terms yield rapidly damped oscillations with period on the order of a and should be discarded in the asymptotic expansion. The asymptotic value of F is defined by the asymptotic value of $h(r)$. The latter is

$$h(r) \approx -\frac{\mu_0 r}{16\pi}\left(1 + \frac{2\gamma}{\varkappa^2 r^2}\right) \tag{2.25}$$

Substitution into (1.24) yields for F

$$F(x) \approx \frac{\mu_0}{r^3} f_1(n) + \frac{\mu_0 \gamma a^2}{r^5} f_2(n) \qquad (2.26)$$

where f_1, f_2 are nondimensional tensor functions of the unit vector $n^\alpha = x^\alpha / r$ and the Poisson coefficient ν. The second member in (2.26) is missing from the Debye model and the asymptotic agrees with the Green's tensor for an elastic continuum. In the general case the second member in (2.26) takes account of the deviation in $\omega(k)$ from a linear law, and includes information on the presence of microstructure in the medium.

In conclusion, let us consider a point defect of pure dilatation type with the moment density

$$m_{\alpha\beta}(x) = v M \delta_{\alpha\beta} \delta(x) \qquad (2.27)$$

as a simple illustration, where M is the relative change in volume of the point defect. Then, according to (1.28) taking account of (2.21) and (2.23), we find for the stress and energy of the defect

$$\sigma^{\alpha\beta}(x) = -\frac{2(1+\nu) vM}{1-\nu} \operatorname{rot}^{\alpha\lambda} \operatorname{rot}_\lambda^{\cdot\beta} \mu(\Delta) g(r) \qquad (2.28)$$

$$\Phi = {}^1/_3 \pi\beta v \mu_0 M^2 \sim v \mu_0 M^2 \qquad (2.29)$$

As $r \to \infty$ the known solution for a center of dilatation results from (2.28).

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CONTACT PROBLEMS OF CREEP THEORY

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The development of creep theory, particularly the proof of the theorem on the influence of creep on the state of stress and strain of an isotropic solid [2 and 26] and the solution of the plane contact problem of plasticity theory [5] produced hypotheses for the analysis of contact problems of creep theory taking account of material ageing. The new effective method of solving first and second kind Fredholm integral equations [18 and 19], which permits obtaining solutions if the solution of the corresponding equation with unit right side is known, also played an essential part. Let us note that from the mechanics viewpoint this solution corresponds to the solution of the plane contact problem for the case of pressure of a rigid stamp with a rectilinear base on a half-plane.

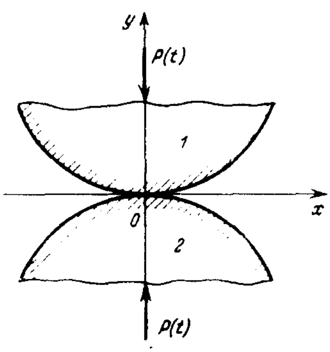

Fig. 1

1. Plane contact problem of creep theory. Prokopovich [26] first studied the plane contact problem of linear creep theory. The known solution of elastisity theory [38] and the fundamental equations [2] of hereditary theory of ageing permitted him to obtain the following Formula :